Math 10B with Professor Stankova
Worksheet, Discussion #20; Thursday, 4/5/2018
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1 Second-Order Linear ODEs

1.1 Concepts

1. Similar to solving second order linear recurrence relations, we guess the solution is of the form λ^n but in this case, we let $\lambda = e^r$ and $n = t$ so we guess the solution is of the form e^{rt}. Then for a ODE $ay'' + by' + cy = 0$, we get the characteristic equation $ar^2 + br + c = 0$ and solve for the roots and the solution is of the form $c_1e^{r_1t} + c_2e^{r_2t}$. If the roots are repeated, the solution is of the form $c_1e^{rt} + c_2te^{rt}$. If the roots are complex of the form $a \pm bi$, then the solution is of the form $c_1e^{at}\sin(bt) + c_2e^{at}\cos(bt)$.

   A **initial value problem** for second linear recurrence relation will tell you $y(0)$ and $y'(0)$. A **boundary value problem** will tell you $y(a), y(b)$, where a, b are the boundaries of the interval you want to find the solution on. An initial value problem will always have a solution. But, a boundary value problem may not have a solution, a unique solution, or infinitely many solutions. It can never have 2, 3, or any finite number other than 0 or 1 solutions.

1.2 Examples

2. Solve the initial value problem $2y'' + 4y' + 2y = 0$ with $y(0) = 0, y'(0) = 1$.

Solution: We guess the solution is of the form $y = e^{rt}$. Plugging this in gives $2r^2e^{rt} + 4re^{rt} + 2e^{rt} = 2e^{rt}(r^2 + 2r + 1) = 0$ and hence $(r + 1)^2 = 0$ so $r = -1$ is a double root. Therefore, the general solution is of the form $y = c_1e^{-t}+c_2te^{-t}$. Plugging in our initial conditions gives $y(0) = 0$ or $0 = c_1e^0 + c_2(0)(e^0) = c_1$ and $y(t) = c_2te^{-t}$ and $y'(t) = c_2t(-e^{-t}) + c_2e^{-t}$ and plugging in $y'(0) = 1$ gives $c_2(0) + c_2(1) = c_2 = 1$ so the solution is $y(t) = te^{-t}$.

3. Solve the boundary value problem of a mass on a spring given by $x'' = -4x$ and $x(0) = 0, x(\pi) = 0$.

Solution: We bring the x's on one side to get $x'' + 4x = 0$ and our characteristic equation is $r^2 + 4 = 0$ with roots $0 \pm 2i$. Therefore, the solution is of the form $x(t) = c_1e^{0t}\cos(2t) + c_2e^{0t}\sin(2t) = c_1\cos(2t) + c_2\sin(2t)$. Plugging in the boundary conditions give $c_1\cos(0) + c_2\sin(0) = c_1 = 0$ and $x(\pi) = c_1\cos(2\pi) + c_2\sin(2\pi) = c_1 = 0$ so the solution is of the form $x(t) = c_2\sin(2t)$. Thus, there are infinitely many solutions.

1.3 Problems

4. True **FALSE** It is possible for there to be no solution to an initial value problem.

5. **TRUE** False It is possible for there to be no solution to a boundary value problem.

6. True **FALSE** All linear ODEs have the property that linear combinations of their solutions are also solutions to them.

Solution: This is only true for linear **homogeneous** ODEs.

7. Solve the initial value problem given by $3y'' = 15y' - 18y$ and $y(0) = 0$ and $y'(0) = 1$.

Solution: We bring all the y's to one side and get $3y'' - 15y' + 18y = 0$ and our characteristic equation is $3r^2 - 15r + 18 = 3(r^2 - 5r + 6) = 3(r-2)(r-3) = 0$ so $r = 2, 3$. So the solution is of the form $y(t) = c_1e^{2t} + c_2e^{3t}$. Plugging in the initial conditions gives $y(0) = c_1 + c_2 = 0$ and $y'(0) = 2c_1 + 3c_2 = 1$ which solving gives $c_2 = 1$ and $c_1 = -1$. Therefore the solution is $y(t) = -e^{2t} + e^{3t}$.

8. Solve the boundary value problem given by $y'' = -y$ and $y(0) = 0, y(\pi) = 1$.

Solution: The differential equation is $y'' + y = 0$ so the characteristic equation is $r^1 + 1 = 0$ or $r = \pm i$. Therefore, the solution is of the form $y(t) = c_1\sin(t) + c_2\cos(t)$. The boundary values give $y(0) = c_2 = 0$ and $y(\pi) = -c_2 = 1$ which cannot happen. Therefore, there are no solutions to this equation.

9. Find the second order linear ODE such that $y(t) = e^{2t}\sin(t)$ is a solution to it.

Solution: Since $e^{2t}\sin(t)$ is a solution, this tells us that the roots are $2 \pm i$. Now in order to find the characteristic equation, we just multiply $(r - (2-i))(r - (2+i)) = r^2 - 4r + 5$. So, the ODE is $y'' - 4y' + 5y = 0$. The initial conditions are $y(0) =, y'(0) = 1$.

10. What is the smallest value of $\alpha > 0$ such that any solution of $y'' + \alpha y' + y = 0$ does not oscillate (does not have any terms of $\sin, \cos$).

> **Solution:** The characteristic equation is given by $r^2 + \alpha r + 1 = 0$. The roots are $\frac{-\alpha \pm \sqrt{\alpha^2 - 4}}{2}$ and this does not have any terms of $\sin, \cos$ whenever $\alpha^2 - 4 \geq 0$ or when $\alpha^2 \geq 4$. Therefore, we must have $\alpha \geq 2$ and the smallest value is $\alpha = 2$.

1.4 Extra Problems

11. Solve the initial value problem $3y'' + 18y' + 27y = 0$ with $y(0) = 0, y'(0) = 1$.

> **Solution:** We guess the solution is of the form $y = e^{rt}$. Plugging this in gives $3r^2e^{rt} + 18re^{rt} + 27e^{rt} = 23e^{rt}(r^2 + 3r + 9) = 0$ and hence $(r + 3)^2 = 0$ so $r = -3$ is a double root. Therefore, the general solution is of the form $y = c_1e^{-3t} + c_2te^{-3t}$. Plugging in our initial conditions gives $y(0) = 0$ or $0 = c_1e^0 + c_2(0)(e^0) = c_1$ and $y(t) = c_2te^{-3t}$ and $y'(t) = c_2t(-3e^{-3t}) + c_2e^{-3t}$ and plugging in $y'(0) = 1$ gives $c_2(0) + c_2(1) = c_2 = 1$ so the solution is $y(t) = te^{-3t}$.

12. Solve the initial value problem given by $2y'' = 3y' - y$ and $y(0) = 0$ and $y'(0) = 1$.

> **Solution:** We bring all the y's to one side and get $2y'' - 3y' + y = 0$ and our characteristic equation is $2r^2 - 3r + 1 = (2r - 1)(r - 1) = 0$ so $r = 1/2, 1$. So the solution is of the form $y(t) = c_1e^{t/2} + c_2e^t$. Plugging in the initial conditions gives $y(0) = c_1 + c_2 = 0$ and $y'(0) = c_1/2 + c_2 = 1$ which solving gives $c_2 = 2$ and $c_1 = -2$. Therefore the solution is $y(t) = -2e^{t/2} + 2e^t$.

13. Solve the boundary value problem given by $y'' + 2y' + 5y = 0$ and $y(0) = 0, y(1) = 1$.

> **Solution:** The characteristic equation is $r^2 + 2r + 5 = 0$ or $r = -1 \pm 2i$. Therefore, the solution is of the form $y(t) = c_1e^{-t}\sin(2t) + c_2e^{-t}\cos(2t)$. The boundary values give $y(0) = c_2 = 0$ and $y(1) = c_1e^{-1}\sin(2) + c_2e^{-1}\cos(2) = 1$. But $c_2 = 0$ and hence $c_1 = e/\sin(2)$ and the solution is given by $y(t) = e/\sin(2)e^{-t}\cos(2t)$.

14. Find the second order linear ODE such that $y(t) = te^{2t}$ is a solution to it.

Solution: Since te^{2t} is a solution, this tells us that 2 is a double root. Now in order to find the characteristic equation, we just multiply $(r-2)^2 = r^2 - 4r + 4$. So, the ODE is $y'' - 4y' + 4y = 0$. The initial conditions are $y(0) = 0, y'(0) = 1$.

15. What is the largest value of $\alpha > 0$ such that any solution of $y'' + 4y' + \alpha y = 0$ does not oscillate (does not have any terms of sin, cos).

Solution: The characteristic equation is given by $r^2 + 4r + \alpha = 0$. The roots are $\frac{-4\pm\sqrt{16-4\alpha}}{2}$ and this does not have any terms of sin, cos whenever $16 - 4\alpha \geq 0$ or when $\alpha \leq 4$. Therefore, the largest value is $\alpha = 4$.

2 Euler's Method

2.1 Concepts

16. Euler's method allows us to approximate solutions to differential equations. Given a differential equation $y' = f(y,t)$ and an initial condition $y(0) = y_0$ and a step size h, we can approximate the path by $y_{n+1} = y_n + f(y_n, t_n)h$. This is gotten by writing $y' = \frac{dy}{dt} \approx \frac{y_{n+1} - y_n}{h}$.

    A slope field is a graph where at every point y, t, you draw a line with the slope there, which is given by the function $f(y,t)$.

2.2 Examples

17. Consider the differential equation $y' = x - y^2$ with initial condition $y(0) = 1$. Use Euler's method to approximate $y(3)$ using step sizes of 1.

Solution: We use the fact that $y' = \frac{dy}{dx} \approx \frac{y(1)-y(0)}{1-0} = 0 - y(0)^2 = -1 = \frac{y(1)-y(0)}{1}$ and hence $y(1) \approx y(0) - 1$. Alternatively, we can write $y(n+1) \approx y(n) + f(n, y(n))(1) = y(n) + n - y(n)^2$. Thus $y(1) \approx 1 + 0 - 1 = 0$ and $y(2) \approx 0 + (1 - 0^2) = 1$ and $y(3) \approx 1 + 3 - 1^2 = 3$.

2.3 Problems

18. **TRUE** False Autonomous equations like $y' = 2\sqrt{y}$ will have slope field that are the same after shifting left and right.

19. **TRUE** False We can only use slope fields and Euler's method when we are given a first order equation.

20. Draw a slope field for $y' = y^2 + x^2$ and sketch the solution when $y(0) = 0$ on the interval $-2 \leq x \leq 2, -2 \leq y \leq 2$.

21. Use Euler's method to estimate $y(3)$ given that $y' = x^2 + y^2$ and $y(0) = 0$ using step sizes of 1.

> **Solution:** We have $y(1) \approx y(0) + (0^2 + 0^2)(1) = 0$ and $y(2) \approx y(1) + (1^2 + 0^2)(1) = 1$ and $y(3) \approx y(2) + (2^2 + 1^2)(1) = 1 + 5 = 6$.

22. Draw a slope field for $y' = y^2 - x^2$ and sketch the solution when $y(0) = 1$ on the interval $0 \leq x \leq 4, 0 \leq y \leq 4$.

23. Use Euler's method to estimate $y(3)$ given that $y' = y^2 - x^2$ and $y(0) = 1$ using step sizes of 1.

> **Solution:** We have $y(1) \approx y(0) + (1^2 - 0^2)(1) = 2$ and $y(2) \approx y(1) + (2^2 - 1^2)(1) = 2 + 3 = 5$ and $y(3) \approx y(2) + (5^2 - 2^2)(1) = 5 + 21 = 26$.